

Lorentz Coherence and the Proton Form Factor

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Abstract

The dipole cutoff behavior for the proton form factor has been and still is one of the major issues in high-energy physics. It is shown that this dipole behavior comes from the coherence between the Lorentz contraction of the proton size and the decreasing wavelength of the incoming photon signal. The contraction rates are the same for both cases. This form of coherence is studied also in the momentum-energy space. The coherence effect in this space can be explained in terms of two overlapping wave functions.

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1 Introduction

Einstein and Bohr met occasionally to discuss physics. Einstein was interested in how things look to moving observers, while Bohr was interested in the electron orbit of the hydrogen atom. Thus, they must have talked about how the orbit looks to a moving observer.

It is possible that they thought about the circular orbit with a longitudinal contraction as illustrated in Fig. 1. This picture of Lorentz contraction is still common in the physics literature [1]. However, after 1927, the orbit became a standing wave. If the standing wave has a rotational symmetry, this figure could still serve a useful purpose. In this paper, we shall see how this figure manifests itself in the Lorentz-covariant quantum world.

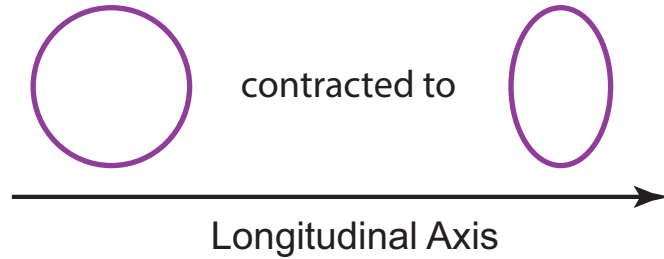


Figure 1: Classical picture of Lorentz contraction. We expect that the longitudinal component becomes contracted while the transverse components are not affected. The issue is how this picture appears in the world of quantum mechanics.

In 1953, Hideki Yukawa was interested in constructing harmonic oscillator wave functions that can be Lorentz-transformed [2]. His primarily interest was in the mass spectrum produced by his Lorentz-invariant differential equation. However, at that time, his mass spectrum did not appear to have anything to do with the physical world.

After witnessing a non-zero charge radius of the proton observed by Hofstadter and McAllister [3], Markov in 1956 considered using Yukawa’s oscillator formalism for calculating the proton form factor [4]. What is the form factor?

However, the constituent particles of the oscillator wave functions were not defined at that time. Shortly after the emergence of the quark model in 1964 [5], Ginzburg and Man’ko in 1965 considered the relativistic harmonic oscillators for bound-state quarks [6].

Using the same harmonic oscillator wave function given in those earlier papers, Fujimura, Kiobayashi, and Namiki calculated the form factor, and concluded it decreases like

$$\frac{1}{(\text{momentum transfer})^4}, \quad (1)$$

for the proton consisting of three spineless quarks [7]. This behavior is called the “dipole cutoff” in the physics literature.

This is a very significant result in view of the fact that this behavior is consistent with what we observe in the real world. In the same year, Licht and Pagnamenta derived the same result using the oscillator Lorentz-contracted wave functions in the Breit coordinate system [8]. Their idea was to by-pass the question of the time-separation variable appearing in the covariant formalism.

In 1971, Feynman, Kislinger, and Ravndal noted that the observed hadronic mass spectra can be explained in terms of the degeneracies of the three-dimensional harmonic oscillators [9], confirming the earlier suggestion made by Yukawa in 1953 [2]. They quoted the paper by Fujimura *et al.* [7], but they did not mention Yukawa's paper. They did not use the normalizable oscillator wave functions developed in those earlier papers.

Their basic problem was that they did not handle the time-separation properly. As the solutions of their Lorentz-invariant differential equation, they contain the Gaussian factor

$$\exp \left\{ - \left(x^2 + y^2 + z^2 - t^2 \right) \right\}. \quad (2)$$

This form is Lorentz-invariant but monotonically increases as the time-separation variable t becomes large. This does not make any sense in physics. Yes, they realize this problem and choose to ignore this variable. In so doing they destroy the mathematical consistency of their paper.

On the other hand Fujimura *et al.* [7] used the Gaussian form

$$\exp \left\{ - \left(x^2 + y^2 + z^2 + t^2 \right) \right\}, \quad (3)$$

without recognizing that it was suggested by Yukawa in 1953 [2], and earlier by Dirac [10]. This form is normalizable in the t variable, but is not invariant under Lorentz transformations. Yet, it can be covariant while this Gaussian function looks differently to a moving observer. Then what role does this t variable play in the covariant formalism?

This problem has been one of my main concerns since I published my first paper on this subject with Marilyn Noz in 1973 [11]. The solution to the problem was to formulate the above-mentioned harmonic oscillator within the framework of Wigner's little groups [12, 13] which dictate the internal space-time symmetries of the particles in the Lorentz-covariant world, while incorporating Dirac's c-number time-energy uncertainty relation [14] and his instant form of relativistic dynamics [15].

According to Wigner's symmetry, the internal space-time symmetry of a massive particle is like $O(3)$ or three-dimensional rotation group. Thus, Feynman *et al.* are justified in ignoring the time-separation variable while concentrating on the three-dimensional space, but they did not do it properly. To make the situation worse, they make an apology of using $O(3)$ instead of $O(3, 1)$. They should not have made this apology.

In the Kim-Noz-Oh paper of 1979 [16], we were able to construct a set of the oscillator wave functions as a representation of Wigner's little group [16]. Earlier in 1977, again in collaboration with Noz, I was able to show that the quark model [5] and the parton model [17, 18] are two different aspects of one Lorentz-covariant entity [19]. This result was reinforced later by my paper of 1989 [20].

The quark and parton models are applicable to two different limits, namely the low-speed and high-speed protons. What happens between those two limits? The proton form factor is indeed a case for studying continuous transition starting from the static proton.

The form factor is a Fourier transformation of the proton density function within the one-photon exchange picture of electron-proton scattering. If the proton wave function is of the Gaussian form as indicated by the hadronic mass spectra, the form factor should be a Gaussian function of the momentum transfer. However, it is not the case in the real world. The decrease is slower, and it is a dipole cutoff as given in Eq.(1).

We are thus led to find the resolution of this problem in the relativistic effect on the size of the proton. Let us consider the scattering of electron and proton with one photon

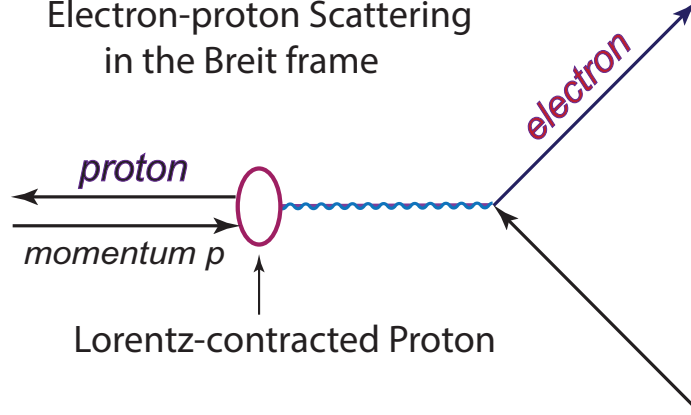


Figure 2: Breit frame. For the scattering of proton and electron with the exchange of one photon, it is possible to choose the Lorentz frame where the momentum of the proton changes to the opposite direction after the collision.

exchange in the Lorentz frame as illustrated in Fig. 2. This frame is known as the Breit frame in the literature.

In this collision process, the total momentum is conserved. The momentum transfer between the particles becomes the momentum of the photon being exchanged. If the momentum transfer becomes larger, the photon wavelength becomes smaller.

As far as the proton is concerned, it receives an external photon signal. As its wavelength becomes smaller, the speed of the proton increases causing a Lorentz contraction of its longitudinal size of the type given in Fig. 1. We thus suspect that there is a “coherence” between the decrease in the wavelength of the photon signal and the width of the Lorentz-contracted proton density. The purpose of this paper is to provide a quantitative analysis for this Lorentz coherence.

In Sec. 2, we study the Lorentz-contraction property of the covariant oscillator wave functions. When boosted, these wave functions become squeezed along the light cones. It is shown that this squeeze leads to Lorentz contraction properties in quantum mechanics. In Sec. 3, we calculate the form factor in detail and study the difference between those with Lorentz-squeezed and with non-squeezed wave functions. It is noted that, as the momentum transfer increases, there is a coherence between the decrease of the wavelength of the incoming photon signal and the contraction of the width of the proton density. This coherence is responsible for the dipole cutoff instead of a steeper Gaussian cutoff. In Sec. 4, we study this coherence problem in the momentum-energy space. The Lorentz coherence appears as an overlap of two squeezed wave functions.

2 Lorentz Contraction of Harmonic Oscillators

Let us consider two quarks bound together by an oscillator potential. If we use x_μ for the space-time separation between the quarks, we can start with the Lorentz-invariant differential equation given by Feynman *et al.* [9]

$$\frac{1}{2} \left\{ - \left(\frac{\partial}{\partial x_\mu} \right)^2 + x_\mu^2 \right\} \psi(x_\mu) = (\lambda + 1) \psi(x_\mu), \quad (4)$$

This equation is separable in the x, y, z , and t coordinates. If the hadron moves along the z direction, the transverse components can be left out. We can concentrate our attention on the two-variable equation

$$\frac{1}{2} \left\{ - \left(\frac{\partial}{\partial z} \right)^2 + \left(\frac{\partial}{\partial t} \right)^2 + z^2 - t^2 \right\} \psi(z, t) = \lambda \psi(z, t). \quad (5)$$

This equation has the solution of the form [2, 10]

$$\frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} (z^2 + t^2) \right\}. \quad (6)$$

This form is not Lorentz-invariant, but remains localized under the Lorentz boost

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}, \quad (7)$$

leading to

$$z \rightarrow z(\cosh \eta) + t(\sinh \eta), \quad \text{and} \quad t \rightarrow z(\sinh \eta) + t(\cosh \eta).$$

Under this transformation, $\psi(z, t)$ becomes

$$\psi_\eta(z, t) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[e^{-2\eta} (z + t)^2 + e^{2\eta} (z - t)^2 \right] \right\}. \quad (8)$$

This wave function is squeezed along the light cones as illustrated in Fig. 3.

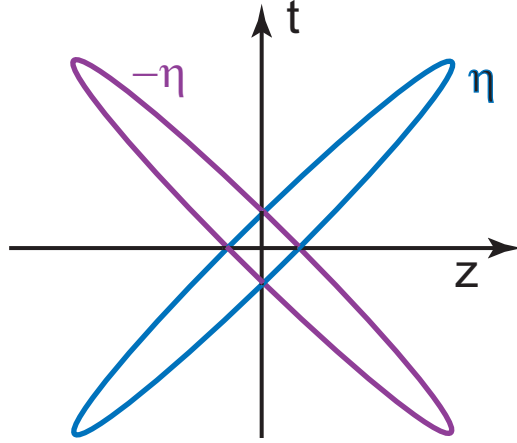


Figure 3: Lorentz-squeezed wave functions. They are squeezed along the light-cones. This figure shows two wave functions moving in opposite directions with the same speed.

The present form of quantum mechanics does not allow time-like excitations [14], while there are excitations along the z direction. Thus, the n -th excited-state wave function is [13]

$$\psi^n(z, t) = \left(\frac{1}{\pi 2^n n!} \right)^{1/2} H_n(z) \exp \left\{ - \left(\frac{z^2 + t^2}{2} \right) \right\}, \quad (9)$$

if the hadron is at rest. If the hadron is Lorentz-boosted, its Lorentz-boosted wave function should take the form

$$\begin{aligned} \psi_\eta^n(z, t) &= \left(\frac{1}{\pi 2^n n!} \right)^{1/2} H_n(z \cosh \eta - t \sinh \eta) \\ &\times \exp \left\{ - \left[\frac{(\cosh 2\eta)(z^2 + t^2) - 4(\sinh 2\eta)zt}{2} \right] \right\}. \end{aligned} \quad (10)$$

Let us consider two wave functions with two different values of n . If two covariant wave functions are in the same Lorentz frame and thus have the same value of η , the orthogonality relation

$$(\psi_\eta^{n'}, \psi_\eta^n) = \delta_{nn'} \quad (11)$$

is satisfied [21]. If those two wave functions have different values of η , we have to start with

$$(\psi_{\eta'}^{n'}, \psi_\eta^n) = \int (\psi_{\eta'}^{n'}(z, t))^* \psi_\eta^n(z, t) dz dt. \quad (12)$$

If $\eta' = 0$, the integral becomes [21]

$$(\psi_0^{n'}, \psi_\eta^n) = \int (\psi_0^{n'}(z, t))^* \psi_\eta^n(z, t) dx dt = \left(\frac{1}{\cosh \eta} \right)^{(n+1)} \delta_{nn'}. \quad (13)$$

It is often more convenient to write the η parameter in terms of $\beta = v/c$, where v is the proton velocity. Then

$$\cosh \eta = \frac{1}{\sqrt{1 - \beta^2}}, \quad e^\eta = \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (14)$$

Then the result of Eq.(13) can be written as

$$(\psi_0^{n'}, \psi_\eta^n) = \left(\sqrt{1 - \beta^2} \right)^{(n+1)} \delta_{nn'}. \quad (15)$$

For the ground state with $n = 0$, it is the Lorentz contraction factor familiar to us. For excited state, the explanation is given in Ref. [13].

It is then not difficult to write the orthogonality relation for the non-zero value of η' , and the result is

$$(\psi_{\eta'}^{n'}, \psi_\eta^n) = \left[\frac{1}{\cosh(\eta - \eta')} \right]^{(n+1)} \delta_{nn'}. \quad (16)$$

With this understanding, hereafter, we deal only with the ground-state wave function and drop the superscript. Thus

$$(\psi_{\eta'}, \psi_\eta) = \frac{1}{\cosh(\eta - \eta')}. \quad (17)$$

Of particular interest is the case with $\eta' = -\eta$, as illustrated in Fig. 3. This means that the two oscillator wave functions are moving in opposite directions. The contraction factor in this case becomes

$$\frac{1}{\cosh(2\eta)} = \frac{1 - \beta^2}{1 + \beta^2}. \quad (18)$$

It is interesting to compare this form with that of $1/\cosh(\eta)$ given in Eq.(14). The difference between these two forms reflect the velocity addition law

$$\frac{2\beta}{1+\beta^2}, \quad (19)$$

for two frames moving in opposite directions with the same speed.

In terms of the momentum variable of the moving hadron p , the contraction factors of Eq.(14) and Eq.(18) can be written as

$$\frac{1}{\cosh(\eta)} = \frac{1}{\sqrt{1+p^2}}, \quad \text{and} \quad \frac{1}{\cosh(2\eta)} = \frac{1}{1+2p^2}, \quad (20)$$

respectively. For simplicity, we use the unit system where the hadronic mass is one. We shall use these formulas to study the proton form factor as a function of p in Sec. 3.

3 Proton Form Factors and Lorentz Coherence

Without recognizing the papers by Yukawa [2], Markov [4], Ginzburg and Man'ko [6], Fujimura *et al.* [7] calculated the electromagnetic form factor of the proton using the oscillator wave function given in those earlier papers. They indeed obtained the desired dipole cutoff.

Also in 1970 [8], Licht and Pagnamenta recognized the problem with the time-separation variable and carried out the same calculation for the scattering system in the Breit frame. In this frame, they were able to by-pass the dependence on the time-separation variable and were able to explain the form factor behavior in terms of the Lorentz-contracted density function.

In the Kim-Noz paper of 1973 [11], we attempted to explain the form factor in terms of the coherence between the incoming signal and the width of the contracted wave function. This aspect was explained also in terms of the overlap of the energy-momentum wave function in our book [13]. In the present paper, I would like to go back to the coherence problem we raised in 1973 [11], and discuss the problem in more detail.

We are considering the scattering one electron and one proton by exchanging one photon. It is then possible to choose the Lorentz frame in which the incoming and outgoing protons are moving in opposite directions with the same speed, as illustrated in Fig. 2. This Lorentz frame is known as the Breit frame.

Let us assume that the proton is moving along the z direction as is indicated in Fig. 2, let p be the magnitude of the momentum, as in the case of Eq.(20). Then their initial and final momentum-energy four-vectors are

$$(p, E) \quad \text{and} \quad (-p, E), \quad (21)$$

respectively, where $E = \sqrt{1+p^2}$. The momentum transfer in this Breit frame is

$$(p, E) - (-p, E) = (2p, 0), \quad (22)$$

with zero energy component.

The form factor then becomes

$$F(p) = \int e^{2ipz} (\psi_\eta(z, t))^* \psi_{-\eta}(z, t) dz dt. \quad (23)$$

If we use the ground-state oscillator wave function, this integral becomes

$$\frac{1}{\pi} \int e^{2ipz} \exp \left\{ -\cosh(2\eta) (z^2 + t^2) \right\} dz dt. \quad (24)$$

The physics of $\cosh(2\eta)$ in this expression was explained in Eq.(18).

In the Fourier integral of Eq.(24), the exponential function does not depend on the t variable. Thus, after the t integration, Eq.(24) becomes

$$F(p) = \frac{1}{\sqrt{\pi \cosh(2\eta)}} \int e^{2ipz} \exp \left\{ -z^2 \cosh(2\eta) \right\} dz. \quad (25)$$

If we complete this integral, the form factor becomes

$$F(p) = \frac{1}{\cosh(2\eta)} \exp \left\{ \frac{-p^2}{\cosh(2\eta)} \right\}. \quad (26)$$

If we use the expression of $\cosh(2\eta)$ given in Eq.(20), this form factor becomes

$$F(p) = \frac{1}{1 + 2p^2} \exp \left(\frac{-p^2}{1 + 2p^2} \right), \quad (27)$$

which decreases as $1/p^2$ for large values of p .

In order to illustrate the effect of the role of this Lorentz contraction in more detail, let us perform the integral of Eq.(25) without the contraction factor $\cosh(2\eta)$. This means that the wave function $\psi_\eta(z, t)$ in the Eq.(23) is replaced by the Gaussian form $\psi_0(z, t)$ of Eq.(6). With this non-squeezed wave function, the Fourier integral becomes

$$G(p) = \int e^{2ipz} (\psi_0(z, t))^* \psi_0(z, t) dz dt. \quad (28)$$

The result of this integral is

$$G(p) = \frac{1}{\sqrt{\pi}} \exp(-p^2). \quad (29)$$

This leads to a Gaussian cutoff of the form factor. This does not happen in the real world, and the calculation of $G(p)$ is for an illustrative purpose only.

Let us go back to the Fourier integrals of Eq.(23) and Eq.(28). The only difference is the $\cosh(2\eta)$ factor in Eq.(23). This factor is in the normalization constant and comes from the integration over the t variable which does not affect the Fourier integral.

However, it causes the Gaussian width to shrink by $1/\sqrt{2}p$ for large values of p . The wave length of sinusoidal factor is inversely proportional to the momentum $2p$. Thus, both the Gaussian width and the wavelength of the incoming signal shrink at the same rate of $1/p$ as p becomes large. Without this coherence, the cutoff is Gaussian as noted in Eq.(29). This effect of this Lorentz coherence is illustrated in Fig. 4.

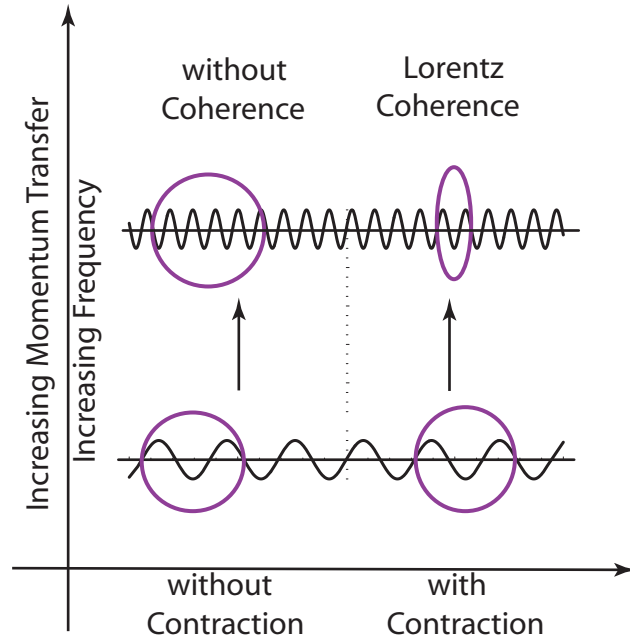


Figure 4: Coherence between the wavelength and the proton size. Let us go to Fig. 2, the proton sees the incoming photon. The wavelength of this photon becomes smaller for increasing momentum transfer. If the proton size remains unchanged, there is a rapid oscillation cutoff in the Fourier integral for the form factor leading to a Gaussian cutoff. However, if the proton size decreases coherently as the wavelength, there are no oscillation effects, leading to a polynomial decrease of the form factor.

There still is a gap between $F(p)$ of Eq.(27) and the real world. Before comparing this expression with the experimental data, we have to realize that there are three quarks inside the proton with two oscillator modes as spelled out in the Appendix.

One of the modes goes through the Lorentz coherence process discussed in this section. The other mode goes through the contraction process given in Eq.(20). The net effect is

$$F_3(p) = \left(\frac{1}{1 + 2p^2} \right)^2 \exp \left(\frac{-p^2}{1 + 2p^2} \right). \quad (30)$$

This will lead to the desired dipole cutoff of $(1/p^2)^2$.

In addition, the effect of the quark spin should be addressed. There are reports of deviations from the exact dipole cutoff. There have been attempts to study the form factors based on the four-dimensional rotation group with imaginary time coordinate. There are also many papers based on the lattice QCD. A brief list of the references to these efforts is given in my recent paper with Marilyn Noz [22].

The purpose of this paper is limited to studying in detail the role of Lorentz coherence in keeping the form factor from the steep Gaussian cutoff in the momentum transfer variable. The coherence problem is one of the primary issues of the current trend in physics.

4 Coherence in Momentum-energy Space

We are now interested in how the Lorentz coherence manifests itself in the momentum-energy space. We can start with the Lorentz-squeezed wave function in the momentum-energy space, which can be written as

$$\phi_\eta(q_z, q_0) = \frac{1}{2\pi} \int e^{-i(q_z z - q_0 t)} \psi_\eta(z, t) dt dz. \quad (31)$$

This is a Fourier transformation of the Lorentz-squeezed wave function of Eq.(8), where q_z and q_0 are Fourier conjugate variables to z and t respectively. The result of this integral is

$$\phi_\eta(q_z, q_0) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[e^{-2\eta} (q_z + q_0)^2 + e^{2\eta} (q_z - q_0)^2 \right] \right\}. \quad (32)$$

In terms of this momentum-energy wave function, the form factor of Eq.(23) can be written as

$$F(p) = \int \phi_{-\eta}^*(q_0, q_z - p) \phi_\eta(q_0, q_z + p) dq_0 dq_z. \quad (33)$$

The evaluation of this integral leads to the form factor $F(p)$ given in Eq.(27).

In order to see the effect of the Lorentz coherence, let us look at two wave functions in Fig. 5. The integral is carried over the q_z q_0 plane. As the momentum p increases, the two wave functions become separated. Without the Lorentz squeeze, the wave functions do not overlap, and this leads to a sharp Gaussian cutoff as in the case of $G(p)$ of Eq.(29).

On the other hand, the squeezed wave functions have an overlap as show in Fig. 5, and this overlap becomes smaller as p increases. This leads to a slower polynomial cutoff [13, 22].

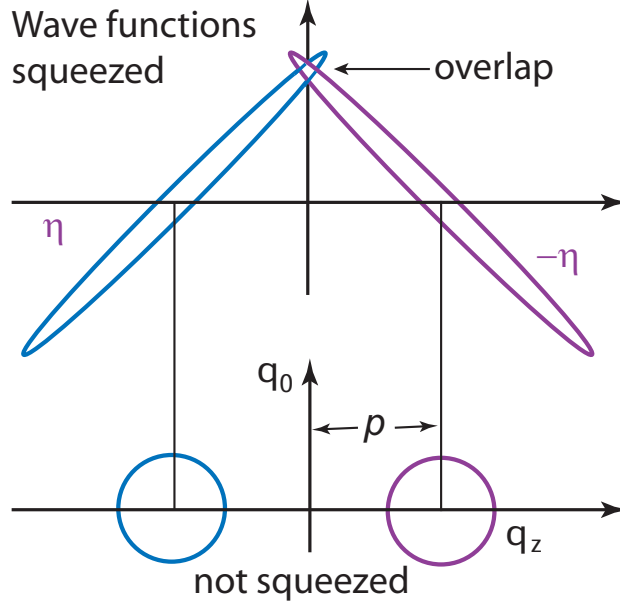


Figure 5: Lorentz coherence in the momentum-energy space. Both squeezed and non-squeezed wave functions are given. As p increases, the two wave functions in Eq.(33) become separated. Without the squeeze, there are no overlaps. This leads to a Gaussian cutoff. The squeezed wave functions maintain an overlap, leading to a slower polynomial cutoff.

Conclusions

Hofstadter's discovery of the non-zero size of the proton opened a new era of physics [3]. The proton is no longer a point particle. One way to measure its internal structure is to study the proton-electron scattering amplitude with one photon exchange, and its dependence on the momentum transfer. The deviation from the case with the point-particle proton is called the proton form factor.

In the experimental front, the dipole cutoff has been firmly established. Yes, there are also experimental results indicating deviations from this dipole behavior [24, 25]. However, in the present paper, no attempts have been made to review all the papers written on the corrections. From the theoretical point of view, those deviations are corrections from the basic dipole behavior.

While the study of the form factor is still a major subject in physics, it is gratifying to note that its dipole cutoff comes from the coherence between the Lorentz contraction of the proton's longitudinal size and the decrease in the wavelength of the incoming signal.

Acknowledgments

Inspired by the paper by Yukawa, Fujimura *et al.*, and Feynman *et al.*, I published my first paper on the Lorentz-covariant harmonic oscillators in 1973 with Marilyn Noz [11]. Shortly after this paper appeared in the Physical Review D, Moisey Markov sent me a reprint of his 1956 review paper published in the Nuovo Cimento [4].

In 1980, I met Vladimir Man'ko in Cocoyoc (Mexico) while attending the 9th Inter-



Figure 6: V. I. Man'ko, Y. S. Kim, and M. A. Markov in 1984 during the 13th International Colloquium on Group Theoretical Methods in Physics held at the University of Maryland.

national Colloquium on Group Theoretical Methods in Physics. He told me he saw my papers on the covariant oscillators and told me about his paper with Ginzburg [6]. In 1984, Vladimir Man'ko came with Moisey Markov to the University of Maryland to attend the 13th the meeting of the same conference. During this conference I had a photo with them as shown in Fig. 6.

In 1986, a graduate student named Yan-Hua Shih of the University of Maryland told me about Horace Yuen's paper on two-photon coherent states [23]. He told me the mathematical formulas in Yuen's paper are very similar to those in my papers on the covariant harmonic oscillators. After examining the paper, I became convinced that the underlying mathematical language for squeezed states of light is that of the Lorentz group. In order to learn more about the subject, I organized in 1991 a workshop on squeezed states at the University of Maryland.

To this conference, I invited five Soviet physicists, including Vladimir Man'ko, Margarita Man'ko, and Victor Dodonov. They became so happy with this conference that they decided to have the second meeting in 1992 at the Lebedev Institute in Moscow. This is how the first meeting on squeezed states became developed into the international conference series known to young physicists these days as the ICSSUR.

I am indeed grateful to Victor Dodonov for inviting me to contribute this paper to this volume dedicated Vladimir Man'ko and Margarita Man'ko on their 75th birthdays. I am fortunate enough to use this occasion to mention Vladimir's early contribution to the subject of harmonic oscillators in the Lorentz-covariant world.

Finally, I am grateful to both Margarita and Vladimir for their ever-lasting cooperation and friendship.

Appendix

Throughout this paper, we used the hadronic system consisting of two quarks bound together by a harmonic oscillator force. The proton however consists of three quarks. In this appendix, we explain how this three-body system becomes that for two oscillators.

This problem was worked out in detail in the 1971 paper of Feynman *et al.* [9]. We choose here to use their notation for the the three quarks. They use u_a, u_b, u_c for the space-time coordinates for those quarks. If there is the oscillator force between the two quarks, we are led to consider the Gaussian form

$$\exp \left\{ -\frac{a}{2} \left[(u_a - u_b)^2 + (u_b - u_c)^2 + (u_c - u_a)^2 \right] \right\}. \quad (34)$$

In order to deal with this form, Feynman *et al.* introduced the following three variables.

$$\begin{aligned} R &= \frac{u_a + u_b + u_c}{3}, \\ x &= \frac{u_b + u_c - 2u_a}{4}, \\ y &= \frac{u_c - u_b}{2\sqrt{3}}, \end{aligned} \quad (35)$$

and

$$\begin{aligned} u_a &= R - 2x, \\ u_b &= R + x - \sqrt{3}y, \\ u_c &= R + x + \sqrt{3}y. \end{aligned} \quad (36)$$

In terms of the variables x and y , the Gaussian function of Eq.(34) becomes

$$\exp \left\{ -\frac{18a}{2} (x^2 + y^2) \right\}. \quad (37)$$

This form does not depend on the coordinate variable R . Thus, the Gaussian form Eq.(34) for the three-body system becomes that for two oscillators.

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